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A general boundary-only formula for crack shape sensitivity of integral functionals

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Abstract. This Note presents, in the framework of three-dimensional linear elastodynamics in the time domain, a method for evaluating crack shape sensitivities of integral functionals, based on the adjoint state approach and resulting in a sensitivity formula expressed in terms of surface integrals (on the external boundary and the crack surface) and contour integrals (involving the direct and adjoint stress intensity factor distributions on the crack front). This method is well-suited to boundary element treatments of e.g. crack reconstruction inverse problems. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

elastodynamics / shape sensitivity / crack / inverse problem / Adjoint state

dérivées de fonctionnelles intégrales dans des perturbations de fissures exprimées par intégrales de frontière

Résumé. *Cette Note propose, dans le cadre de l'elastodynamique linéaire tridimensionnelle en variable temporelle, une méthode d'évaluation de la sensibilité de fonctionnelles intégrales à des perturbations de fissures, qui repose sur la définition d'un état adjoint et permet d'exprimer la sensibilité en termes d'intégrales de surface (frontière externe et fissure) et de contour (contribution des distributions directes et adjointes de facteurs d'intensité de contraintes sur le front de fissure). Ce résultat est par exemple bien adaptée aux techniques de reconstruction de fissures fondées sur les éléments de frontière. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

elastodynamique / variation de domaine / fissure / problème inverse / état adjoint

Version française abrégée

Evaluer la sensibilité de fonctionnelles intégrales par rapport à des perturbations du domaine est notamment utile dans des situations à domaine inconnu (problèmes inverses), ajustable (optimisation) ou variable. Cette Note propose une formulation de la sensibilité à des perturbations de fissures, développée dans le cadre de l'elastodynamique linéaire tridimensionnelle en variable temporelle et fondée sur la méthode de l'état adjoint.

On considère ainsi un corps élastique borné $\Omega \in \mathbf{R}^3$, de frontière externe S et contenant une fissure Γ. Les déplacements $\boldsymbol{u}$, déformations $\boldsymbol{\varepsilon}$ et contraintes $\boldsymbol{\sigma}$ vérifient les équations de champ (1) et les conditions aux limites et initiales (2), l'ensemble constituant le *problème direct*.

Note présentée par Huy Duong BUI

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La sensibilité aux perturbations de fissure de fonctionnelles intégrales de la forme (3) est analysée en considérant des transformations de domaine qui laissent S fixe, c.à.d. de la forme $\boldsymbol{x}^\eta = \boldsymbol{x} + \eta\boldsymbol{\theta}(\boldsymbol{x})$ où η est un paramètre cinématique et la vitesse de transformation $\boldsymbol{\theta}(\boldsymbol{x})$ vérifie $\boldsymbol{\theta} = \boldsymbol{0}$ sur S. Si $\overset{\star}{f} = f_{,\eta} + \boldsymbol{\nabla} f.\boldsymbol{\theta}$ désigne la dérivée lagrangienne de f, les dérivées d'intégrales génériques sont données par (4).

Adoptant les méthodes du contrôle optimal, le problème direct est introduit comme contrainte explicite dans le lagrangien (5), dont la dérivée dans une perturbation de fissure est trouvée égale à (6). Notre choix de dérivation lagrangienne résulte du fait que la dérivée partielle $(\boldsymbol{\nabla} u)_{,\eta}$ est singulière comme $d^{-3/2}$ le long du front $\partial\Gamma$ tandis que $\boldsymbol{\nabla} \overset{\star}{\boldsymbol{u}}$ et $\boldsymbol{\nabla}\boldsymbol{u}$ ont la même singularité $d^{-1/2}$ (d : distance à $\partial\Gamma$). Ensuite, l'état adjoint $\tilde{\boldsymbol{u}}_\Gamma$, choisi de façon à annuler les termes en $\overset{\star}{\boldsymbol{u}}$ et $\overset{\star}{\boldsymbol{p}}$, est défini par les conditions aux limites et *finales* (8). Cela conduit à l'expression (9) de la sensibilité. Cette dernière expression pourrait être immédiatement convertie en intégrales de frontières à l'aide de l'identité (10) en l'absence de singularités en front de fissure. Ici, il est nécessaire d'isoler un voisinage tubulaire $D_\varepsilon = \{\boldsymbol{x}, \text{dist}(\boldsymbol{x}, \partial\Gamma) \leq \varepsilon\}$ du front $\partial\Gamma$, d'utiliser les expressions bien connues (12) des parties singulières des champs et d'évaluer la contribution non nulle de l'intégrale sur Σ_ε quand $\varepsilon \to 0$. On obtient ainsi, résultat principal de cette Note, l'expression (13) de la sensibilité en termes d'intégrales de surface et de contour.

L'équation (13) s'applique à des fissures et des vitesses de transformation régulières assez générales. Elle est par exemple bien adaptée aux techniques de reconstruction de fissures fondées sur les éléments de frontière. Soulignons qu'elle concerne des *perturbation de configurations fixes de fissures*, non des propagations. Bien entendu, elle s'applique, moyennant les modifications évidentes, à l'éastostatique et l'élastodynamique en domaine fréquentiel. Par exemple, $\mathcal{J}(\Gamma)$ est l'énergie potentielle à l'équilibre pour le choix particulier $\varphi_u = -(\boldsymbol{p}^D.\boldsymbol{u})/2$, $\varphi_p = (\boldsymbol{u}^D.\boldsymbol{p})/2$ dans Eq. (3), l'état adjoint étant alors $\tilde{\boldsymbol{u}} = (1/2)\boldsymbol{u}$, soit $\tilde{K}_I = K_I/2$, etc. Dans (13), le facteur de $(\boldsymbol{\theta}.\boldsymbol{\nu})(s)$ est bien, comme attendu, l'opposé du taux de restitution d'énergi $G(s)$, tandis que le facteur de $(\boldsymbol{\theta}.\boldsymbol{n})(s)$ est l'équivalent 3-D de l'intégrale J_2 [4, 5]. L'intégrale H [6] appara^it également comme un cas particulier de (13).

1. Introduction

The need to compute the sensitivity of integral functionals with respect to shape parameters arises in many situations where a geometrical domain plays a primary role; shape optimization and inverse problems are the most obvious, as well as possibly the most important, of such instances. In addition to numerical differentiation techniques, shape sensitivity evaluation can be based on either direct differentiation or the adjoint variable approach, this paper being focused on the latter.

The object of this Note is to present a method for evaluating crack shape sensitivities of integral functionals, in the framework of three-dimensional linear elastodynamics in the time domain. This method is based on the adjoint state approach and results in a sensitivity formula expressed in terms of surface integrals (on the external boundary and the crack surface) and contour integrals (involving the direct and adjoint stress intensity factor distributions on the crack front). This sensitivity formula is thus well-suited to the use of boundary element methods (BEMs), which are quite often used for solving e.g. crack reconstruction inverse problems, see e.g. [11, 13].

Consider an elastic body $\Omega \in \mathbf{R}^3$ of finite extension, externally bounded by the closed surface S and containing a crack Γ. The displacement $\boldsymbol{u}$, strain ε and stress $\boldsymbol{\sigma}$ are related by the field equations:

$$\text{div}\,\boldsymbol{\sigma} - \rho\ddot{\boldsymbol{u}} = \boldsymbol{0} \qquad \boldsymbol{\sigma} = \boldsymbol{C}\!:\!\varepsilon \qquad \varepsilon = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u}) \qquad \text{in } \Omega \tag{1}$$

($\boldsymbol{C}$: fourth-order elasticity tensor). Besides, displacements and tractions are prescribed on the portions S_u and $S_p = S \setminus S_u$ of S, the crack surface Γ is stress-free and initial rest is assumed:

$$\boldsymbol{u} = \boldsymbol{u}^D \ (\text{on } S_u), \qquad \boldsymbol{p} = \boldsymbol{p}^D \ (\text{on } S_p), \qquad \boldsymbol{p} = \boldsymbol{0} \ (\text{on } \Gamma), \qquad \boldsymbol{u} = \dot{\boldsymbol{u}} = \boldsymbol{0} \ (\text{in } \Omega, \text{at } t = 0) \tag{2}$$

where $\boldsymbol{p} \equiv \boldsymbol{\sigma}.\boldsymbol{n}$ is the traction vector, defined in terms of the outward unit normal $\boldsymbol{n}$ to Ω. The above conditions define the *direct problem*.

Let us introduce the following generic objective function:

$$\mathcal{J}(\Gamma) = J(\boldsymbol{u}_\Gamma, \boldsymbol{p}_\Gamma, \Gamma) = \int_0^T \int_{S_p} \varphi_u(\boldsymbol{u}_\Gamma, \boldsymbol{x}, t)\,\mathrm{d}S\,\mathrm{d}t + \int_0^T \int_{S_u} \varphi_p(\boldsymbol{p}_\Gamma, \boldsymbol{x}, t)\,\mathrm{d}S\,\mathrm{d}t + \int_\Gamma \psi(\boldsymbol{x})\,\mathrm{d}S \qquad (3)$$

which is encountered for instance in minimization-based algorithms for solving the inverse problem of crack detection ($(\boldsymbol{u}_\Gamma, \boldsymbol{p}_\Gamma)$ refer to the solution of problem (1, 2) for a given crack configuration). A boundary-only expression for the derivative of $\mathcal{J}(\Gamma)$ with respect to crack perturbations is sought.

Any sufficiently small perturbation of Γ can be described by means of a domain transformation which does not affect the external boundary S, i.e. of the form $\boldsymbol{x}^\eta = \boldsymbol{x} + \eta\boldsymbol{\theta}(\boldsymbol{x})$ where η is a time-like parameter and the transformation velocity field $\boldsymbol{\theta}(\boldsymbol{x})$ is such that $\boldsymbol{\theta} = \boldsymbol{0}$ on S. Denoting by $\overset{\star}{f} = f_{,\eta} + \boldsymbol{\nabla} f.\boldsymbol{\theta}$ the Lagrangian derivative of some field variable f, the derivatives at $\eta = 0$ of integrals over generic domains V and surfaces Σ have the well-known form:

$$\frac{d}{d\eta}\int_V f\,\mathrm{d}V = \int_V (\overset{\star}{f} + f\,\mathrm{div}\,\boldsymbol{\theta})\,\mathrm{d}V \qquad \frac{d}{d\eta}\int_\Sigma f\,\mathrm{d}S = \int_\Sigma (\overset{\star}{f} + f\,\mathrm{div}_S\boldsymbol{\theta})\,\mathrm{d}S \qquad (4)$$

where $\mathrm{div}_S\boldsymbol{\theta} = \mathrm{div}\,\boldsymbol{\theta} - \boldsymbol{n}.\boldsymbol{\nabla}\boldsymbol{\theta}.\boldsymbol{n}$ is the surface divergence of $\boldsymbol{\theta}$ ($\boldsymbol{n}$: outward unit normal vector). Also, recall that $(\boldsymbol{\nabla}\boldsymbol{u})^\star = \boldsymbol{\nabla}\overset{\star}{\boldsymbol{u}} - \boldsymbol{\nabla}\boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{\theta}$.

2. Adjoint problem and domain integral formulation

For the present purposes, it is convenient to use an optimal control approach, whereby the variables in the objective function $J(\boldsymbol{u}, \boldsymbol{p}, \Gamma)$ are formally considered as independent ones and the direct problem (i.e. the fact that one actually has $\boldsymbol{u} = \boldsymbol{u}_\Gamma, \boldsymbol{p} = \boldsymbol{p}_\Gamma$) is treated as an explicit constraint. The following Lagrangian is thus introduced:

$$\begin{aligned}\mathcal{L}(\boldsymbol{u}, \tilde{\boldsymbol{u}}, \boldsymbol{p}, \tilde{\boldsymbol{p}}, \Gamma) = J(\boldsymbol{u}, \boldsymbol{p}, \Gamma) &+ \int_0^T \int_\Omega [\boldsymbol{\sigma} : \boldsymbol{\nabla}\tilde{\boldsymbol{u}} + \rho\ddot{\boldsymbol{u}}.\tilde{\boldsymbol{u}}]\,\mathrm{d}V\,\mathrm{d}t \\ &- \int_0^T \int_{S_u} (\boldsymbol{u} - \boldsymbol{u}^D).\tilde{\boldsymbol{p}}\,\mathrm{d}S\,\mathrm{d}t - \int_0^T \int_{S_u} \boldsymbol{p}.\tilde{\boldsymbol{u}}\,\mathrm{d}S\,\mathrm{d}t - \int_0^T \int_{S_p} \boldsymbol{p}^D.\tilde{\boldsymbol{u}}\,\mathrm{d}S\,\mathrm{d}t \qquad (5)\end{aligned}$$

where $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$, the test functions of the direct problem in weak form, act as Lagrange multipliers.

Using formulas (4), noticing that $(\boldsymbol{C} : \boldsymbol{\nabla}\overset{\star}{\boldsymbol{u}}) : \boldsymbol{\nabla}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{\sigma}} : \boldsymbol{\nabla}\overset{\star}{\boldsymbol{u}}$ and ignoring the terms containing $\overset{\star}{\tilde{\boldsymbol{u}}}, \overset{\star}{\tilde{\boldsymbol{p}}}$ arising in this calculation (they merely reproduce the direct problem constraint and thus vanish), the total derivative of $\mathcal{L}$ with respect to a given domain perturbation is given by:

$$\begin{aligned}\overset{\star}{\mathcal{L}}(\boldsymbol{u}, \tilde{\boldsymbol{u}}, \boldsymbol{p}, \tilde{\boldsymbol{p}}, \Gamma) = &\int_0^T \int_\Omega [\tilde{\boldsymbol{\sigma}} : \boldsymbol{\nabla}\overset{\star}{\boldsymbol{u}} + \rho\,\overset{\star}{\ddot{\boldsymbol{u}}}.\tilde{\boldsymbol{u}}]\,\mathrm{d}V\,\mathrm{d}t \\ &- \int_0^T \int_{S_u} \Big(\tilde{\boldsymbol{u}} - \frac{\partial\varphi_p}{\partial\boldsymbol{p}}\Big).\overset{\star}{\boldsymbol{p}}\,\mathrm{d}S\,\mathrm{d}t - \int_0^T \int_{S_u} \tilde{\boldsymbol{p}}.\overset{\star}{\boldsymbol{u}}\,\mathrm{d}S\,\mathrm{d}t + \int_0^T \int_{S_p} \frac{\partial\varphi_u}{\partial\boldsymbol{u}}.\overset{\star}{\boldsymbol{u}}\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega \Big\{[\boldsymbol{\sigma} : \boldsymbol{\nabla}\tilde{\boldsymbol{u}} + \rho\ddot{\boldsymbol{u}}.\tilde{\boldsymbol{u}}]\mathrm{div}\,\boldsymbol{\theta} - [\boldsymbol{\sigma}.\boldsymbol{\nabla}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\sigma}}.\boldsymbol{\nabla}\boldsymbol{u}] : \boldsymbol{\nabla}\boldsymbol{\theta}\Big\}\,\mathrm{d}V\,\mathrm{d}t \\ &+ \int_\Gamma [\boldsymbol{\nabla}\psi.\boldsymbol{\theta} + \psi\,\mathrm{div}_S\boldsymbol{\theta}]\,\mathrm{d}S \qquad (6)\end{aligned}$$

The partial derivative $(\boldsymbol{\nabla}u)_{,\eta}$ has generally a $d^{-3/2}$ singularity along the crack edge $\partial\Gamma$, while $\boldsymbol{\nabla}\overset{\star}{\boldsymbol{u}}$ and $\boldsymbol{\nabla}\boldsymbol{u}$ have the same $d^{-1/2}$ singularity (d: distance to $\partial\Gamma$); this explains our using Lagrangian derivatives $\overset{\star}{\boldsymbol{u}}$.

Since the initial conditions $\boldsymbol{u}(\cdot,0)=\dot{\boldsymbol{u}}(\cdot,0)=\mathbf{0}$ hold for any location of the assumed defect, one should assume $\overset{\star}{\boldsymbol{u}}(\cdot,0)=\dot{\overset{\star}{\boldsymbol{u}}}(\cdot,0)=\mathbf{0}$ as well. Eq. (6) is thus altered by taking into account the identity:

$$\int_0^T \ddot{\overset{\star}{\boldsymbol{u}}}.\tilde{\boldsymbol{u}}\,\mathrm{d}t=(\dot{\overset{\star}{\boldsymbol{u}}}.\tilde{\boldsymbol{u}}-\dot{\tilde{\boldsymbol{u}}}.\overset{\star}{\boldsymbol{u}})\,|_{t=T}+\int_0^T \overset{\star}{\boldsymbol{u}}.\ddot{\tilde{\boldsymbol{u}}}\,\mathrm{d}t \tag{7}$$

Now, the multipliers $\tilde{\boldsymbol{u}},\tilde{\boldsymbol{p}}$ are chosen specifically so that all terms containing $\overset{\star}{\boldsymbol{u}}$ and $\overset{\star}{\boldsymbol{p}}$ in Eq. (6) combine to zero for any $\overset{\star}{\boldsymbol{u}}$ and $\overset{\star}{\boldsymbol{p}}$. The weak formulation of an *adjoint problem*, of a form similar to the direct problem in (5) but with $\overset{\star}{\boldsymbol{u}},\overset{\star}{\boldsymbol{p}}$ now acting as test functions, is thus defined. Its *adjoint solution* $\tilde{\boldsymbol{u}}_\Gamma,\tilde{\boldsymbol{p}}_\Gamma$ is therefore found (by analogy to (5) to solve equations (1) together with the following boundary and *final* conditions:

$$\tilde{\boldsymbol{p}}=-\frac{\partial\varphi_u}{\partial\boldsymbol{u}}\ \ (\text{on } S_p)\qquad \tilde{\boldsymbol{u}}=\frac{\partial\varphi_p}{\partial\boldsymbol{p}}\ \ (\text{on } S_u)\qquad \tilde{\boldsymbol{p}}=\mathbf{0}\ \ (\text{on } \Gamma^\pm)\qquad \tilde{\boldsymbol{u}}=\dot{\tilde{\boldsymbol{u}}}=0\ \ (\text{in } \Omega, \text{at } t=T) \tag{8}$$

Finally, Eq. (6) allows to express the derivative of J in terms of the direct and adjoint solutions:

$$\begin{aligned}\overset{\star}{\mathcal{J}}(\Gamma)&=\overset{\star}{\mathcal{L}}(\boldsymbol{u}_\Gamma,\tilde{\boldsymbol{u}}_\Gamma,\boldsymbol{p}_\Gamma,\tilde{\boldsymbol{p}}_\Gamma,\Gamma)\\&=\int_0^T\!\!\int_\Omega\Big\{[\boldsymbol{\sigma}_\Gamma:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma+\rho\ddot{\boldsymbol{u}}_\Gamma.\tilde{\boldsymbol{u}}_\Gamma]\mathrm{div}\,\boldsymbol{\theta}-[\boldsymbol{\sigma}_\Gamma.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma+\tilde{\boldsymbol{\sigma}}_\Gamma.\boldsymbol{\nabla}\boldsymbol{u}_\Gamma]:\boldsymbol{\nabla}\boldsymbol{\theta}\Big\}\,\mathrm{d}V\,\mathrm{d}t\\&\quad+\int_\Gamma[\boldsymbol{\nabla}\psi.\boldsymbol{\theta}+\psi\mathrm{div}_S\boldsymbol{\theta}]\,\mathrm{d}S\end{aligned} \tag{9}$$

3. Sensitivity in terms of boundary integrals

The next step is to find an equivalent form of Eq. (9) involving only boundary integrals. Any elastodynamic states $(\boldsymbol{u},\boldsymbol{\sigma})$ and $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\sigma}})$ satisfying initial and final rest conditions, respectively, verify the identity:

$$\begin{aligned}&\int_0^T\Big\{[\boldsymbol{\sigma}:\tilde{\varepsilon}+\rho\ddot{\boldsymbol{u}}.\tilde{\boldsymbol{u}}]\mathrm{div}\,\boldsymbol{\theta}-[\boldsymbol{\sigma}.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{\sigma}}.\boldsymbol{\nabla}\boldsymbol{u}]:\boldsymbol{\nabla}\boldsymbol{\theta}\Big\}\,\mathrm{d}t\\&\qquad=\int_0^T\mathrm{div}\Big([\boldsymbol{\sigma}:\tilde{\varepsilon}-\rho\dot{\boldsymbol{u}}.\dot{\tilde{\boldsymbol{u}}}]\boldsymbol{\theta}-[\boldsymbol{\sigma}.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{\sigma}}.\boldsymbol{\nabla}\boldsymbol{u}].\boldsymbol{\theta}\Big)\,\mathrm{d}t\end{aligned} \tag{10}$$

However, the well-known singular behaviour of strains and stresses at the crack front prevent a direct application of the divergence formula to Eq. (9) for the entire cracked domain Ω. To circumvent this difficulty, the body Ω is partitioned into $\Omega=\Omega_\varepsilon\cup(D_\varepsilon\setminus\Gamma)$, where $D_\varepsilon=\{\boldsymbol{x},\mathrm{dist}(\boldsymbol{x},\partial\Gamma)\le\varepsilon\}$ for some sufficiently small $\varepsilon>0$ is a tubular neighbourhood of the crack front $\partial\Gamma$ bounded by the tubular surface Σ_ε. Further, put $\Omega_\varepsilon=\Omega\setminus\bar{D}_\varepsilon$ and $\Gamma_\varepsilon=\Gamma\setminus\bar{D}_\varepsilon$. Upon introducing this splitting into Eq. (9), applying the divergence formula for the contribution of Ω_ε and invoking boundary conditions (2_3, 8_3), one obtains:

$$\begin{aligned}\overset{\star}{\mathcal{J}}(\Gamma)&=\int_0^T\!\!\int_{\Sigma_\varepsilon}\Big\{[\boldsymbol{\sigma}_\Gamma:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma-\rho\dot{\boldsymbol{u}}_\Gamma.\dot{\tilde{\boldsymbol{u}}}_\Gamma](\boldsymbol{\theta}.\boldsymbol{n})-[\boldsymbol{p}_\Gamma.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma+\tilde{\boldsymbol{p}}_\Gamma.\boldsymbol{\nabla}\boldsymbol{u}_\Gamma].\boldsymbol{\theta}\Big\}\,\mathrm{d}S\,\mathrm{d}t\\&\quad+\int_0^T\!\!\int_{\Gamma_\varepsilon}[\![\boldsymbol{\sigma}_\Gamma:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma-\rho\dot{\boldsymbol{u}}_\Gamma.\dot{\tilde{\boldsymbol{u}}}_\Gamma]\!](\boldsymbol{\theta}.\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t\\&\quad+\int_0^T\!\!\int_{D_\varepsilon}\Big\{[\boldsymbol{\sigma}_\Gamma:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma+\rho\ddot{\boldsymbol{u}}_\Gamma.\tilde{\boldsymbol{u}}_\Gamma]\mathrm{div}\,\boldsymbol{\theta}-[\boldsymbol{\sigma}_\Gamma.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma+\tilde{\boldsymbol{\sigma}}_\Gamma.\boldsymbol{\nabla}\boldsymbol{u}_\Gamma]:\boldsymbol{\nabla}\boldsymbol{\theta}\Big\}\,\mathrm{d}V\,\mathrm{d}t\\&\quad+\int_\Gamma[\boldsymbol{\nabla}\psi.\boldsymbol{\theta}+\psi\mathrm{div}_S\boldsymbol{\theta}]\,\mathrm{d}S\end{aligned} \tag{11}$$

where $\boldsymbol{n}$ is the outward unit normal to Ω_ε and $[\![f]\!] \equiv f(\boldsymbol{x}^+) - f(\boldsymbol{x}^-)$ (discontinuity of f across Γ).

Now, the limiting form when $\varepsilon \to 0$ of Eq. (11) is sought. Recall the well-known expansions of the mechanical fields near the *fixed* crack front, isotropic elasticity being assumed (ν: Poisson ratio, μ: shear modulus):

$$\begin{aligned}
u_r &= \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\Big[K_I(s,t)\cos\frac{\theta}{2}(3-4\nu-\cos\theta) + K_{II}(s,t)\sin\frac{\theta}{2}(4\nu-1+3\cos\theta)\Big] + O(r) \\
&= u_r^S(r,\theta,s) + O(r) \\
u_\theta &= \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\Big[-K_I(s,t)\sin\frac{\theta}{2}(1-4\nu-3\cos\theta) + K_{II}(s,t)\cos\frac{\theta}{2}(4\nu-5+3\cos\theta)\Big] + O(r) \\
&= u_\theta^S(r,\theta,s) + O(r) \\
u_z &= \frac{2K_{III}(s,t)}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2} + O(r) = u_z^S(r,\theta,s) + O(r)
\end{aligned} \tag{12}$$

and similarly for $\tilde{\boldsymbol{u}}$ with stress intensity factors $\tilde{K}_{I,II,III}$; (r,θ) denote local polar coordinates, attached to a point $\boldsymbol{x}(s)$ of $\partial\Gamma$ characterized by its arc length s, in the plane orthogonal to $\partial\Gamma$ and emanating from $\boldsymbol{x}(s)$, and z is such that (r,θ,z) define cylindrical coordinates. The singular displacements defined by Eqs. (12) produce direct and adjoint strains and stresses, denoted $(\boldsymbol{\varepsilon}^S,\boldsymbol{\sigma}^S)$ and $(\tilde{\boldsymbol{\varepsilon}}^S,\tilde{\boldsymbol{\sigma}}^S)$. Since by virtue of these expansions $\boldsymbol{\sigma}:\tilde{\boldsymbol{\varepsilon}} = O(1/r)$, the integral over D_ε vanishes in the limit ($\mathrm{d}V = r(1+O(r))\,\mathrm{d}r\mathrm{d}\theta\,\mathrm{d}s$ in D_ε). Besides, it can be verified that $[\![\boldsymbol{\sigma}^S:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^S]\!] = O(d)$, and hence that the integral over Γ_ε becomes in the limit $\varepsilon \to 0$ the corresponding, convergent, integral over Γ. Finally, under mild smoothness assumptions on the closed curve $\partial\Gamma$ and the velocity field $\boldsymbol{\theta}$, one has:

$$\begin{aligned}
&\int_0^T\int_{\Sigma_\varepsilon}\Big\{[\boldsymbol{\sigma}_\Gamma:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma - \rho\dot{\boldsymbol{u}}_\Gamma.\dot{\tilde{\boldsymbol{u}}}_\Gamma](\boldsymbol{\theta}.\boldsymbol{n}) - [\boldsymbol{p}_\Gamma.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}_\Gamma + \tilde{\boldsymbol{p}}_\Gamma.\boldsymbol{\nabla}\boldsymbol{u}_\Gamma].\boldsymbol{\theta}\Big\}\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad = \int_0^T\int_{\partial\Gamma}\int_{-\pi}^{\pi}\Big\{[\boldsymbol{\sigma}^S:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^S](\boldsymbol{\theta}.\boldsymbol{n})(s) - [\boldsymbol{p}^S.\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^S + \tilde{\boldsymbol{p}}^S.\boldsymbol{\nabla}\boldsymbol{u}^S].\boldsymbol{\theta}(s)\Big\}\varepsilon\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}t + O(\varepsilon^{1/2})
\end{aligned}$$

The integral in the right-hand side, which yields a finite contribution as the radius ε of the tubular neighbourhood goes to zero, can be evaluated in a straightforward way using expansions (12). This last calculation results in the following expression of $\overset{\star}{\mathcal{J}}(\Gamma)$, which constitutes the main result of this Note:

$$\begin{aligned}
\overset{\star}{\mathcal{J}}(\Gamma) = &\int_\Gamma(\boldsymbol{\theta}.\boldsymbol{n})(s)\int_0^T [\![\boldsymbol{\sigma}:\boldsymbol{\nabla}\tilde{\boldsymbol{u}} - \rho\dot{\boldsymbol{u}}.\dot{\tilde{\boldsymbol{u}}}]\!]\,\mathrm{d}t\,\mathrm{d}S \\
&- \frac{1}{\mu}\int_{\partial\Gamma}(\boldsymbol{\theta}.\boldsymbol{\nu})(s)\int_0^T\Big\{(1-\nu)[K_I\tilde{K}_I + K_{II}\tilde{K}_{II}] + K_{III}\tilde{K}_{III}\Big\}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&+ \frac{1-\nu}{\mu}\int_{\partial\Gamma}(\boldsymbol{\theta}.\boldsymbol{n})(s)\int_0^T(K_I\tilde{K}_{II} + K_{II}\tilde{K}_I)(s,t)\,\mathrm{d}t\,\mathrm{d}s
\end{aligned} \tag{13}$$

where $\boldsymbol{\nu}(s)$ denotes the unit outward normal to $\partial\Gamma$ lying in the tangent plane to Γ at $\boldsymbol{x}(s)$. To ensure that formula (13) can actually be evaluated using only the boundary traces of the direct and adjoint solutions, the bilinear form $\boldsymbol{\sigma}:\boldsymbol{\nabla}\tilde{\boldsymbol{u}}$ must be expressed in terms of $\boldsymbol{\nabla}_S\boldsymbol{u}$, $\boldsymbol{\nabla}_S\tilde{\boldsymbol{u}}$, taking $\boldsymbol{p} = \tilde{\boldsymbol{p}} = \boldsymbol{0}$ into account in the process:

$$\boldsymbol{\sigma}:\boldsymbol{\nabla}\tilde{\boldsymbol{u}} = \mu\Big\{\frac{2\nu}{1-\nu}\mathrm{div}_S\boldsymbol{u}\;\mathrm{div}_S\tilde{\boldsymbol{u}} + \frac{1}{2}(\boldsymbol{\nabla}_S\boldsymbol{u} + \boldsymbol{\nabla}_S^T\boldsymbol{u}):(\boldsymbol{\nabla}_S\tilde{\boldsymbol{u}} + \boldsymbol{\nabla}_S^T\tilde{\boldsymbol{u}}) - (\boldsymbol{n}.\boldsymbol{\nabla}_S\boldsymbol{u}).(\boldsymbol{n}.\boldsymbol{\nabla}_S\tilde{\boldsymbol{u}})\Big\} \tag{14}$$

Discussion

Equation (13) holds true for general smooth crack shapes and transformation velocity fields. It provides an attractive computational tool, in conjunction with BEMs and gradient-based optimization algorithms, for solving inverse or optimisation problems for unknown or adjustable cracks. Both the direct and the adjoint problems stated in this Note can be solved by usual BEM techniques [3], including the evaluation of the direct and adjoint SIF distributions [10]. Other applications of shape sensitivity analysis using the adjoint state approach in connexion with BEMs can be found in e.g. [1, 2, 7, 8, 9, 12]. The present approach is certainly applicable to integral functionals somewhat more general than those of the form (3). Integral functionals defined in terms of domain integrals over the body Ω still lead to the boundary-only sensitivity expression (13) but body force distributions appear in the definition of the adjoint problem.

It is important to stress that Eq. (13) provides the sensitivity of an integral functional to a *perturbation of a fixed crack configuration*, not a crack propagation, hence the use of expansions (12), valid for a crack which does not physically propagate.

Equation (13) is also applicable, with straightforward modifications, to elastostatics and elastodynamics in the frequency domain. For instance, in elastostatics, $\mathcal{J}(\Gamma)$ is the potential energy at equilibrium for the particular choice $\varphi_u = -(\boldsymbol{p}^D.\boldsymbol{u})/2$, $\varphi_p = (\boldsymbol{u}^D.\boldsymbol{p})/2$ in Eq. (3). For this special case, the adjoint solution turns out to be $\tilde{\boldsymbol{u}} = (1/2)\boldsymbol{u}$, i.e. $\tilde{K}_I = K_I/2$, etc. In equation (13), the factor of $(\boldsymbol{\theta}.\boldsymbol{\nu})(s)$ turns out to be, as expected, minus the energy release rate $G(s)$, i.e. minus the J_1-integral, whereas the factor of $(\boldsymbol{\theta}.\boldsymbol{n})(s)$ is the 3-D generalization of the J_2-integral [4, 5]. Finally, with the choice $S_p = S, s_u = \emptyset$ and $\varphi_p = \boldsymbol{p}^D.\hat{\boldsymbol{u}} - \boldsymbol{u}.\hat{\boldsymbol{p}}$, where $\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}$ are the boundary traces of a *pre-selected* auxiliary elastodynamic state with *final* homogeneous conditions, one finds that $\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}}$ and that the factor of $(\boldsymbol{\theta}.\boldsymbol{\nu})(s)$ in (13) is the 3-D generalization of the so-called H-integral [6].

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